International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 7, 2022, pp. 287-308
Received 30 Apr 2022. Published on-line 27 Jul 2022
web: http://www.journal-1.eu/

Euler Coordinates in the Plane of a Triangle

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Abstract. Euler coordinates (x, y) for the plane of a triangle ABC, not including the line at infinity, are introduced and studied using barycentric coordinates. The Euler coordinate system has the Euler line as x-axis and the orthic axis as y-axis. The origin, $(0, 0)$, is the point of intersection of the two axes, which are orthogonal. Among topics discussed using Euler coordinates are distances, lines, circles and circle-inversions, conics, reflections, Shinagawa coefficients, and a transformation T defined on ordered pairs (P, U) of distinct points. Much of the material presented here was computer-discovered, including a section on special properties of the orthic axis.

Keywords. triangle geometry, Euler line, orthic axis, barycentric coordinates, Shinagawa coefficients

Mathematics Subject Classification (2020). 51-04, 51N20, 39B22.

1. Introduction

We begin with a few historical notes about a standard way of labeling a triangle and the Euler line, as shown in Figure 1. According to Florian Cajori [1], it was Euler who "simplified formulas by the simple expedient of designating the angles of a triangle by A, B, C, and the opposite sides by a, b, c, respectively, and only once before have we encountered this simple device." Actually, in the earlier occurrence, the roles of the lower case letters and the capitals were reversed; this was in a pamphlet prepared by Richard Rawlinson at Oxford sometime between 1655 and

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1668.[3] About a century later, Euler proved that the centroid, circumcenter, and orthocenter of ABC are collinear. Their line is now known as the Euler line. Although those three points were known to Euclid, there appears to be no record of their collinearity prior to Euler's proof, written in 1763 and published in 1765. For details, see [5], [6], and [7].

Although Cartesian coordinates had been in place since the time of Descartes (1596-1637), it was not until 1827, more than half a century after Euler's collinearity proof, that homogeneous barycentric coordinates were introduced, by Augustus Ferdinand Möbius (1790-1868).

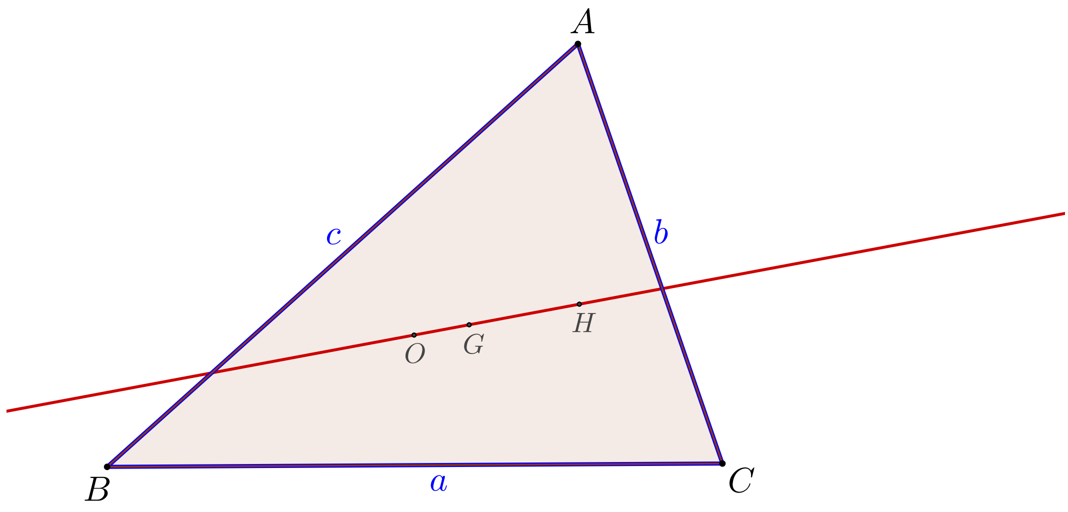

FIGURE 1. Triangle ABC sidelengths a, b, c, and the Euler line.

Basic facts about homogeneous barycentric coordinates (henceforth *barycentrics*) are given in Yiu [12]. Associated with barycentrics are *normed barycentrics* (also called *normalized barycentrics*): if a point P in the plane of ABC has barycentrics $p : q : r$ and P does not lie on the line at infinity, then the normed barycentrics of P are the ordered triple (p^*, q^*, r^*) of signed areas of the triangles BPC, CPA, APB, respectively, given by

$$p^* = \frac{p}{p+q+r}, \quad q^* = \frac{q}{p+q+r}, \quad r^* = \frac{r}{p+q+r}.$$

In the sequel, we write (p, q, r) instead of (p^*, q^*, r^*). The distinction between (ordinary homogeneous) barycentrics and normed barycentrics is that the former, as ratios, are written with colons (and without parentheses), whereas the notation for normed barycentrics is simply the standard notation for an ordered triple.

[3]Rawlinson's pamphlet has been identified [6] as a "Short Trigonometry", of which the only known copy is in the British Library. The earliest known labeling of a triangle using A, B, C, a, b, c appears on page 573.

TABLE 1. Selected triangle centers and barycentrics

$X_1 = I =$ incenter	$a : b : c$
$X_2 = G =$ centroid	$1 : 1 : 1$
$X_3 = O =$ circumcenter	$a^2(b^2 + c^2 - a^2) ::$
$X_4 = H =$ orthocenter	$1/(b^2 + c^2 - a^2) ::$
$X_{30} =$ Euler infinity point	$2a^4 - (b^2 - c^2)^2 - a^2(b^2 + c^2) ::$
$X_{230} =$ (see below)	$a^2(2a^2 - b^2 - c^2) - (b^2 - c^2)^2 ::$
$X_{468} =$ (see below)	$(b^2 + c^2 - 2a^2)/(b^2 + c^2 - a^2) ::$
$X_{647} =$ (see below)	$a^2(b^2 - c^2)(b^2 + c^2 - a^2) ::$

Descriptive names of the points $X_{230}, X_{468}, X_{647}$ in [4] are as follows:

$$X_{230} = X_2\text{-Ceva conjugate of } X_{114};$$
$$X_{468} = X_2\text{-line conjugate of } X_3;$$
$$X_{647} = \text{crossdifference of } X_2 \text{ and } X_3.$$

We shall use Conway triangle notation [11]:

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2};$$
$$S^2 = S_B S_C + S_C S_A + S_A S_B = 4(\text{area of } ABC)^2$$
$$S_\omega = S \cot \omega = \frac{a^2 + b^2 + c^2}{2} = S_A + S_B + S_C, \text{ where } \omega = \text{ Brocard angle of } ABC.$$

We shall also use these additional functions that are symmetric in a, b, c :

$$E = \frac{(S_B + S_C)(S_C + S_A)(S_A + S_B)}{S^2};$$
$$F = \frac{S_A S_B S_C}{S^2};$$
$$N = \sqrt{E - 8F} = 6|GO|, \text{ so that } N > 0;$$
$$R = \frac{abc}{2S} = \text{the circumradius of } ABC;$$
$$\mathcal{L}^\infty = \text{the line at infinity, given by } x + y + z = 0.$$

Regarding $\mathcal{L}^\infty$, it is assumed throughout this article that the notations P and U refer to points that are not on $\mathcal{L}^\infty$.

Triangle centers will be represented as in [4] and also by capital letters for certain classical points, included in Table 1. Every triangle center X has barycentrics of the form $f(a, b, c) : f(b, c, a) : f(c, a, b)$, so that X can be represented more concisely as $f(a, b, c) ::$, as in Table 1.

2. EULER COORDINATES

The Euler coordinate system introduced here for the plane of a triangle ABC has as orthogonal axes the Euler line and the orthic axis, labeled as the x-axis and the y-axis, respectively. These axes meet in X_{468}. The Euler coordinates for a point P are written in the same form as Cartesian coordinates: (x, y), where

$$x = x'N \quad \text{and } y = y'NS, \tag{1}$$

where

(2) $$x' = \text{signed distance from } P \text{ to the } y\text{-axis;}$$

(3) $$y' = \text{signed distance from } P \text{ to the } x\text{-axis.}$$

Here, "signed distance" has the same meaning as in the Cartesian coordinate system, depending on definitions of positive x-axis and y-axis, as follows: the positive x-axis consists of all points on the Euler line that lie on the same side of the origin, $(0,0) = X_{468}$, as G. The positive y-axis consists of all points on the orthic axis that lie on the same side of $(0,0)$ as the point

$$P^+ = p^+ : q^+ : r^+ = (0, S_\omega^2)$$

given by

(4) $$p^+ = (E+F)S_B S_C - 3FS^2 + (S_B - S_C)S_\omega^2,$$

where q^+ and r^+ are defined cyclically. (The rationale for this choice is given in Remark 1 in Section 3.) A mate for P^+ that lies on the negative y-axis for every ABC is given by

(5) $$p^- = (E+F)S_B S_C - 3FS^2 - (S_B - S_C)S_\omega^2,$$

where q^- and r^- are defined cyclically. Although points P^+ and P^- are not triangle centers, they are a bicentric pair indexed as P_{201} and U_{201} in [3].

We may now regard the plane of ABC as consisting for four quadrants, as in Cartesian coordinates; see Figure 2.[4]

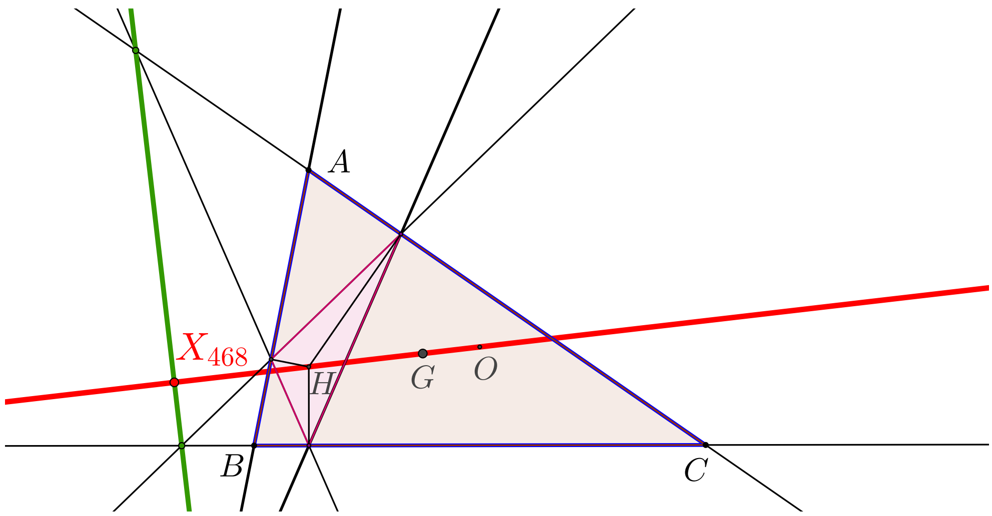

FIGURE 2. Euler coordinate system, Euler line (red), orthic axis (green)

The orthic axis, orthogonal to the Euler line as in Figure 2, is constructed as the perspectrix of two perspective triangles: orthic and ABC.

Using coordinates shown in Table 2, we have the following theorem and corollary:

Theorem 2.1. *The centroid, G, has Euler coordinates* $\big((a^2+b^2+c^2)/6, 0\big)$.

[4]In Figure 2, the positive y-axis is the $\pi/2$-counterclockwise rotation of the positive x-axis. However, if the configuration is reflected in the y-axis, the positive y-axis is no longer the $\pi/2$-counterclockwise rotation of the positive x-axis. This shows that orientation does not suffice for defining the positive y-axis.

Proof. The Euler coordinates of G are $(x, y) = ((E+F)/3, 0)$, where

$$\begin{aligned} x &= \frac{(S_B+S_C)(S_C+S_A)(S_A+S_B)}{3S^2} + \frac{S_AS_BS_C}{3S^2} \\ &= \frac{(a+b+c)(a^2+b^2+c^2)(b+c-a)(c+a-b)(a+b-c)}{24S^2} \\ &= \frac{a^2+b^2+c^2}{6}. \end{aligned}$$

In keeping with our choice of G to define the positive direction on the x-axis, Theorem 2.1 confirms that for $G = (x, 0)$, we really do have $x > 0$ for all a, b, c. □

Corollary 2.1. *In the plane of every triangle ABC, the point X_{23} (the circumcircle-inverse of X_2) lies on the negative x-axis.*

Proof. Since $X_{23} = (-E-F, 0)$, the fact that $x < 0$ follows from the positiveness of $E + F$. □

In Table 2 and thereafter, the notation $\prec f(a,b,c) \succ$ abbreviates the following cyclic sum:

$$f(a,b,c) + f(b,c,a) + f(c,a,b).$$

TABLE 2. Coordinates: barycentric and Euler

X_n	barycentrics	(x, y)
X_1	$a : b : c$	$\left(\frac{\prec aS_A \succ}{\prec a \succ}, \frac{\prec aS_A(S_B-S_C)\succ}{\prec a \succ}\right)$
X_2	$1 : 1 : 1$	$((E+F)/3, 0)$
X_3	$S^2 - S_BS_C ::$	$((E-2F)/2, 0)$
X_4	$S_BS_C ::$	$(3F, 0)$
X_5	$S^2 + S_BS_C ::$	$((E+4F)/4, 0)$
X_6	$S_B + S_C ::$	$\left(\frac{S^2}{E+F}, -\frac{\prec S_A^2(S_B-S_C)\succ}{2(E+F)}\right)$
X_{23}	$(E+4F)S^2 - 4(E+F)S_BS_C ::$	$(-E-F, 0)$
X_{468}	$\frac{b^2+c^2-2a^2}{b^2+c^2-a^2} ::$	$(0, 0)$
X_{647}	$a^2(b^2-c^2)(b^2+c^2-a^2) ::$	$\left(0, \frac{2(E+F)^2FS^2-(E-2F)S^4}{\prec S_A^2(S_B-S_C)\succ}\right)$

3. Distances

The distance between points $P = (p, q, r)$ and $U = (u, v, w)$ is given [12] by

$$|PU|^2 = S_A(p-u)^2 + S_B(q-v)^2 + S_C(r-w)^2. \tag{6}$$

Lemma 3.1. *The orthic axis is given by*

$$S_A\alpha + S_B\beta + S_C\gamma = 0. \tag{7}$$

Proof. It is easy to check (by computer) that X_{230} and X_{231}, both on the orthic axis, satisfy (7). □

Lemma 3.2. *Suppose that $x = x(a, b, c)$ is symmetric in a, b, c and homogeneous of degree 2. Then the equation*

$$S_A\alpha + S_B\beta + S_C\gamma = x \tag{8}$$

represents a line that is parallel to the orthic axis.

Proof. Using a formula for parallel lines (e.g., [12], p. 44), it is easy to check that the line (8) is parallel to the line (7) and passes through the following point on the Euler line:

$$\begin{aligned}&(2a^2-b^2-c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)-2\big(a^2(2a^2-b^2-c^2)-(b^2-c^2)^2\big)x ::\\&=(E+F)S_BS_C-3FS^2+(S^2-3S_BS_C)x ::\end{aligned}$$

□

Two examples illustrating Lemma 3.2 follow. The line

$$S_A\alpha+S_B\beta+S_C\gamma=S_\omega/3,$$

parallel to the orthic axis, passes through the centroid, G. The line

$$S_A\alpha+S_B\beta+S_C\gamma=a^2+b^2+c^2,$$

also parallel to the orthic axis, passes through X_{46517}.

Lemma 3.3. *The Euler line is given by*

$$S_A(S_B-S_C)\alpha+S_B(S_C-S_A)\beta+S_C(S_A-S_B)\gamma=0. \tag{9}$$

Proof. It is easy to check that G and O satisfy this equation. □

Lemma 3.4. *Suppose that $y=y(a,b,c)$ is symmetric in a,b,c and homogeneous of degree 4 in a,b,c. Then the equation*

$$S_A(S_B-S_C)\alpha+S_B(S_C-S_A)\beta+S_C(S_A-S_B)\gamma=y \tag{10}$$

represents a line that is parallel to the Euler line.

Proof. It is easy to check that the line (10) is parallel to the line (9) and passes through the following point on the orthic axis:

$$\begin{aligned}&(2a^2-b^2-c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)-4(b^2-c^2)y ::\\&=(E+F)S_BS_C-3FS^2+(S_B-S_C)y ::\end{aligned}$$

□

By (8) and (10), the point $P=(p,q,r)$ that has Euler coordinates (x,y) satisfies the matrix equation

$$M\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}1\\x\\y\end{pmatrix} \tag{11}$$

where

$$M=\begin{pmatrix}1&1&1\\S_A&S_B&S_C\\S_A(S_B-S_C)&S_B(S_C-S_A)&S_C(S_A-S_B)\end{pmatrix}, \tag{12}$$

so that

$$\begin{pmatrix}p\\q\\r\end{pmatrix}=M^{-1}\begin{pmatrix}1\\x\\y\end{pmatrix}. \tag{13}$$

Explicitly,

$$p = \frac{S_B S_C(S_B + S_C - 2S_A) + (S_A S_B + S_C S_A - 2S_B S_C)x + (S_B - S_C)y}{S_A^2(S_B + S_C) + S_B^2(S_C + S_A) + S_C^2(S_A + S_B) - 6S_A S_B S_C} \tag{14}$$

$$= \frac{(E+F)S_B S_C - 3FS^2 + (S^2 - 3S_B S_C)x + (S_B - S_C)y}{N^2 S^2}, \tag{15}$$

and q and r are determined cyclically.

Remark 1. *Let (x, y) be the Euler coordinates of a point*

$$P = p(a, b, c) : q(a, b, c) : r(a, b, c) = p(a, b, c) : p(b, c, a) : p(c, a, b).$$

By (15), after multiplying by N^2S^2, we have the first barycentric p represented as

$$p = (E + F)S_B S_C - 3FS^2 + (S^2 - 3S_B S_C)x + (S_B - S_C)y. \tag{16}$$

Now if $y > 0$ for all a, b, c, then by (16), we have $p(a, b, c) = p(a, c, b)$. Consequently, P is not a triangle center. Instead, the points (x, y) and $(x, -y)$ comprise a bicentric pair, and, regarding triangle centers, we conclude that no triangle center stays on the same side of the Euler line for all triangles ABC.

Lemma 3.5. *Let P_x be the point of intersection of the lines (8) and (9). Barycentrics for P_x are given by*

$$P_x = (E + F)S_B S_C - 3FS^2 + (S^2 - 3S_B S_C)x :: . \tag{17}$$

Proof. This follows from (15) with $y = 0$. □

Theorem 3.1. *Let $P = (x, y)$ and let d_x be the signed distance between P and the orthic axis. Then $x = Nd_x$.*

Proof. Using (17) and (6), we have

$$\begin{aligned} N^4S^4d_x^2 &= S_A\big((E+F)S_BS_C - 3FS^2 + x(S^2 - 3S_BS_C - (E+F)S_BS_C + 3FS^2)\big)^2 \\ &\quad + S_B\big((E+F)S_CS_A - 3FS^2 + x(S^2 - 3S_CS_A - (E+F)S_CS_A + 3FS^2)\big)^2 \\ &\quad + S_C\big((E+F)S_AS_B - 3FS^2 + x(S^2 - 3S_AS_B - (E+F)S_AS_B + 3FS^2)\big)^2 \\ &= x^2\big(S_A(S^2 - 3S_BS_C)^2 + S_B(S^2 - 3S_CS_A)^2 + S_C(S^2 - 3S_AS_B)^2\big) \\ &= x^2(E - 8F)S^4, \end{aligned}$$

so that $x = Nd_x$. □

Lemma 3.6. *Let P_y be the point of intersection of the lines (7) and (10) in the case that y is symmetric in a, b, c. Then barycentrics for P_y are given by*

$$P_y = \frac{(E + F)S_B S_C - 3FS^2 + (S_B - S_C)y}{N} :: . \tag{18}$$

Proof. This follows from (15) with $x = 0$. □

Theorem 3.2. *Let $P = (x, y)$ and let d_y be the signed distance between P and the Euler line. Then $y = NSd_y$.*

Proof. Again using (17) and (6),

$$\begin{aligned} N^4S^4d_y^2 &= S_Ay^2(S_B-S_C)^2 + S_By^2(S_C-S_A)^2 + S_Cy^2(S_A-S_B)^2 \\ &= y^2(E-8F)S^2, \end{aligned}$$

so that $y = NSd_y$. □

4. FORMULAS AND EXAMPLES

This section presents several formulas and examples, some of which are matched to Mathematica programs in the Appendix.

4.1. **From barycentrics to Euler coordinates.** If P is a point with normed barycentrics (p, q, r), then Euler coordinates (x, y) for P are given by (8) and (10) as follows:

$$x = S_Ap + S_Bq + S_Cr; \tag{19}$$

$$y = S_A(S_B-S_C)p + S_B(S_C-S_A)q + S_C(S_A-S_B)r. \tag{20}$$

4.2. **From Euler coordinates to barycentrics.** If P has Euler coordinates (x, y), then normed barycentrics (p, q, r) for P are given by (15). Consequently, homogeneous barycentrics for P are as shown here:

$$p : q : r = (E+F)S_BS_C - 3FS^2 + (S^2 - 3S_BS_C)x + (S_B-S_C)y :: \ . \tag{21}$$

In (21), the coordinate x must be of degree 2 in a, b, c, and y, of degree 4, as stipulated in Lemmas 3.2 and 3.4.

4.3. **Lines.** The line of $P = p : q : r$ and $U = u : v : w$ is given by

$$(qw-rv)\alpha + (ru-pw)\beta + (pv-qu)\gamma = 0. \tag{22}$$

If the Euler coordinates for P and U are (x_1, y_1) and (x_2, y_2), where $x_1 = x_2$, then the line (22) is given by

$$y = \frac{y_2-y_1}{x_2-x_1}(x-x_1) + y_1. \tag{23}$$

Equation (23) is the same as in Cartesian coordinates, so that formulas for vertical lines, intersections, parallelism, midpoints, reflections, and orthogonality are already familiar.

As an example, the Brocard axis passes through the circumcenter and the symmedian point:

$$X_3 = ((E-2F)/2, 0); \tag{24}$$

$$X_6 = \left(\frac{S^2}{E+F}, -\frac{\prec S_A^2(S_B-S_C) \succ}{2(E+F)}\right), \tag{25}$$

so that the Brocard axis is represented in Euler coordinates by

$$y = m\left(x - \frac{E-2F}{2}\right), \text{ where } m = \frac{\prec S_A^2(S_B-S_C) \succ}{(E+F)(E-2F) - 2S^2}. \tag{26}$$

The intersection of the Brocard axis and the orthic axis is obtained from (26) with $x = 0$:

$$\begin{aligned}(x, y) &= (0, -\frac{(E-2F) \prec S_A^2(S_B - S_C) \succ}{2(E+F)(E-2F) - 4S^2} \\ &= (E+F)(E-2F) - 3S^2 + 3S_BS_C - S_A^2 + 3FS_A :: \\ &= X_{3003}.\end{aligned}$$

4.4. **Distances.** As already noted in Section 3, the distance d between $P = (p, q, r)$ and $U = (u, v, w)$ is given by

$$d^2 = S_A(p-u)^2 + S_B(q-v)^2 + S_C(r-w)^2, \tag{27}$$

and if P and U have Euler coordinates (x_1, y_1) and (x_2, y_2), then

$$d^2 = \frac{(x_1 - x_2)^2S^2 + (y_1 - y_2)^2}{N^2S^2}. \tag{28}$$

4.5. **Circles.** By (28), the circle with center (x_1, y_1) and radius r is given by

$$\frac{(x - x_1)^2S^2 + (y - y_1)^2}{N^2S^2} = r^2. \tag{29}$$

Consequently, the circle with center (x_1, y_1) and pass-through point (x_2, y_2) is given by

$$(x - x_1)^2S^2 + (y - y_1)^2 = (x_2 - x_1)^2S^2 + (y_2 - y_1)^2. \tag{30}$$

Table 3 shows several circles whose representation in Euler coordinates is much shorter than in barycentrics.

TABLE 3. Selected Circles

center	pass-through point	equation
$(0, 0) = X_{468}$	G	$9S^2x^2 + 9y^2 = (E+F)^2$
$(0, 0) = X_{468}$	O	$4S^2x^2 + 4y^2 = (E-2F)^2$
$(0, 0) = X_{468}$	H	$S^2x^2 + y^2 = 9F^2$
$(0, 0) = X_{468}$	X_{23}	$S^2x^2 + y^2 = (E+F)^2$
$O = X_3$	$(0, 0)$	$S^2(x - \frac{E-2F}{2})^2 + y^2 = S^2(\frac{E-2F}{2})^2$
$O = X_3$	$X(k)$ for $k = 98, \ldots, 112$	$S^2(x - \frac{E-2F}{2})^2 + y^2 = N^2S^2R^2$
X_{381}	X_2, X_4	$S^2(x - \frac{E+10F}{6})^2 + y^2 = \frac{S^2N^4}{36}$

Note that Table 3 includes, in the last two rows, the circumcircle and the orthocentroidal circle.

As an example using an equation in Table 3, we find that the Brocard axis, given by (26), intersects the circumcircle in the points X_{1379} and X_{1380} with Euler coordinates (x, y) given by

$$\begin{aligned}x &= \frac{E-2F}{2} \pm \frac{NSR}{\sqrt{m^2+S^2}}; \\ y &= \pm\frac{mNSR}{\sqrt{m^2+S^2}}.\end{aligned}$$

4.6. **Inversion in a circle.** Let (O) denote an arbitrary circle, given by (29), with center $U = (x_0, y_0)$, and suppose that $P = (x_P, y_P)$ is a point not on (O) such that the slope

$$m = \frac{y_P - y_0}{x_P - x_0}$$

of the line UP exists and is not 0. The (O)-inverse of P is defined as the point P' on UP satisfying $|OP||OP'| = r^2$. We seek formulas for the coordinates (x', y') of P'. Substituting

$$y_P = m(x_P - x_0) + y_0 \quad \text{and} \quad y' = m(x' - x_0) + y_0$$

in

$$|OP|^2 = \frac{(x_P - x_0)^2 S^2 + (y_P - y_0)^2}{N^2 S^4} \quad \text{and} \quad |OP'|^2 = \frac{(x' - x_0)^2 S^2 + (y' - y_0)^2}{N^2 S^4}$$

leads to

$$|x' - x_0| = \frac{S^4 N^2 r^2}{(S^2 + m^2)|x_P - x_0|}.$$

It follows that if P lies outside (O), then

$$x' = x_0 + \frac{S^4 N^2 r^2}{(S^2 + m^2)(x_P - x_0)},$$

and that if P lies insides (O), then

$$x' = x_0 - \frac{S^4 N^2 r^2}{(S^2 + m^2)(x_P - x_0)}.$$

Suppose next that for the same circle (O), the point $P = (x_0, y_P)$, so that there is no slope m. Then the (O)-inverse of P is the point (x_0, y') given by

$$y' = \begin{cases} y_0 + \frac{m^2}{m^2+S^2} \frac{S^4 N^2 r^2}{y_P - y_0} & \text{if } P \text{ lies outside } (O); \\ \\ y_0 - \frac{m^2}{m^2+S^2} \frac{S^4 N^2 r^2}{y_P - y_0} & \text{if } P \text{ lies inside } (O). \end{cases}$$

On the other hand, if $P = (x_P, y_0)$, so that $m = 0$, then the (O)-inverse of P is the point (x', y_0) given by

$$x' = \begin{cases} x_0 + \frac{S^2 N^2 r^2}{x_P - x_0} & \text{if } P \text{ lies outside } (O); \\ \\ x_0 - \frac{S^2 N^2 r^2}{x_P - x_0} & \text{if } P \text{ lies inside } (O). \end{cases}$$

As an example, the circumcircle-inverse of $X_5 = ((E + 4F)/4, 0)$ is the point $X_{2070} = (-(E + 2F)/2, 0)$, and conversely.

4.7. **Conics.** Here, we find equations in Euler coordinates for parabolas, ellipses, and hyperbolas in standard position, with the point $X_{468} = (0, 0)$ as center. To begin, let $\mathcal{P}$ be the parabola with focus $F = (0, k)$ and directrix $y = -k$. If $P = (x, y)$ is a point on $\mathcal{P}$, then equating $|PF|$ to the distance between P and the point $(x, -k)$ gives

$$\frac{x^2 S^2 + (y - k)^2}{S^4 N^2} = \frac{(y + k)^2}{S^4 N^2}, \tag{31}$$

which simplifies to

$$x^2 S^2 = 4ky.$$

Likewise, if $F = (h, 0)$, then the directrix is $x = -h$, and the parabola is given by

$$y^2 = 4S^2hx. \tag{32}$$

For $X = (x, y)$ on the ellipse or hyperbola with center $(0, 0)$ and foci $F_1 = (f, 0)$ and $F_2 = (-f, 0)$, we have, for some $k > 0$,

$$\frac{\left|\sqrt{(x-f)^2S^2 + y^2} \pm \sqrt{(x+f)^2S^2 + y^2}\right|}{NS^2} = k. \tag{33}$$

Squaring both sides and simplifying,

$$\begin{aligned} &\pm 2\sqrt{(x-f)^2S^2 + y^2}\sqrt{(x+f)^2S^2 + y^2} \\ &= k^2N^2S^4 - (x-f)^2S^2 - (x+f)^2S^2 - 2y^2. \end{aligned}$$

Squaring and simplifying again gives

$$\frac{4}{k^2N^2S^2}x^2 - \frac{4}{(4f^2 - k^2N^2S^2)S^2}y^2 = 1, \tag{34}$$

which represents an ellipse or hyperbola according as $kNS > 2f$ or $kNS < 2f$. Putting $y = 0$ in (34) shows that the vertices are

$$(\pm\frac{kNS}{2}, 0).$$

Note that translations are natural in Euler coordinates. In particular, translating a conic along the Euler line is easily accomplished by a substitution $x \to x - x_0$ in (32) or (34).

4.8. **Reflections.** Reflections about the x- and y- axes in barycentrics can be very lengthy but are straightforward in Euler coordinates:

$$\begin{aligned} (x, -y) &= \text{reflection of } (x, y) \text{ in } x\text{-axis;} \\ (-x, y) &= \text{reflection of } (x, y) \text{ in } y\text{-axis;} \\ (-x, -y) &= \text{reflection of } (x, y) \text{ in origin.} \end{aligned}$$

Reflections in Euler coordinates and reflections in Cartesian coordinates are symbolically identical, but care must be taken with regard to distances; e.g., the distance between $(x, 0)$ and $(x', 0)$ is not $|x - x'|$, but $|x - x'|/N$; the distance between points $(0, y)$ and $(0, y')$ is $|y - y'|/(NS)$.

A formula for the reflection of an arbitrary point $P = (x_0, y_0)$ in a nonvertical line can be derived as in a Cartesian coordinate system, as follows. Let the line, L, be represented by $y = y' + m(x - x')$, so that the line $L^\perp$ through P perpendicular to L is given by $y = y_0 - S^2(x - x_0)/m$. Then the point $(x'', y'') = L \cap L^\perp$ is given by

$$x'' = \frac{my_0 + S^2x_0 - my' + m^2x'}{m^2 + S^2}, \quad y'' = y' + m(x'' - x'). \tag{35}$$

The desired reflection, denoted by (x^*, y^*), is given by

$$x^* = 2x'' - x_0 = \frac{x_0 - m^2x_0 + 2m^2x' + 2my_0 - 2my'}{m^2 + S^2};$$
$$y^* = 2y'' - y_0 = \frac{2m^2S^2x_0 - 2mS^2x' - S^2y_0 + m^2y_0 + 2S^2y'}{m^2 + S^2}.$$

5. RECTANGLES

Suppose that P is a point with Euler coordinates (x, y), where $xy = 0$. Then the reflection of P in the y-axis has Euler coordinates $(-x, y)$, and, likewise, as in Cartesian coordinates, other reflections in the coordinate axes and the origin are easily represented. If P is a triangle center, then the various reflections that yield vertices of rectangles are also triangle centers. Figure 3 shows such a configuration of rectangles (without scaling), corresponding to $P = X_{46980}$.

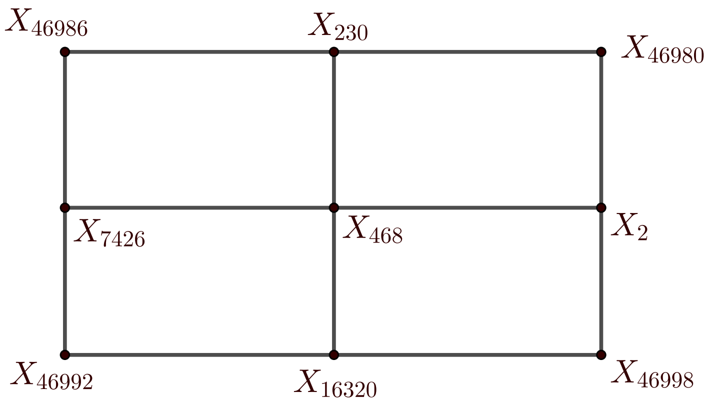

FIGURE 3. Rectangle with center $(0, 0)$ and vertex X_{46980}, with midpoints.

In addition to the rectangles in Figure 3 we present twelve more such configurations:

Rectangle with center $(0, 0)$ and vertex X_{46981}

$$\begin{array}{ccc} X_{46987} & X_{230} & X_{46981} \\ X_{11799} & X_{468} & X_3 \\ X_{46993} & X_{16320} & X_{46999} \end{array}$$

Rectangle with center $(0, 0)$ and vertex X_{46982}

$$\begin{array}{ccc} X_{46988} & X_{230} & X_{46982} \\ X_{10295} & X_{468} & X_4 \\ X_{46994} & X_{16320} & X_{47000} \end{array}$$

Rectangle with center $(0, 0)$ and vertex X_{46983}

$$\begin{array}{ccc} X_{46989} & X_{647} & X_{46983} \\ X_{7426} & X_{468} & X_2 \\ X_{46995} & X_{47004} & X_{47001} \end{array}$$

Rectangle with center $(0, 0)$ and vertex X_{46984}

$$\begin{array}{ccc} X_{46990} & X_{647} & X_{46984} \\ X_{7799} & X_{468} & X_3 \\ X_{46996} & X_{47004} & X_{47002} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{46985}

$$\begin{array}{ccc} X_{46991} & X_{647} & X_{46985} \\ X_{10295} & X_{468} & X_{4} \\ X_{46997} & X_{47004} & X_{47003} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_1

$$\begin{array}{ccc} X_{11809} & X_{16272} & X_{1} \\ X_{47346} & X_{468} & X_{3109} \\ X_{47321} & X_{16309} & X_{47270} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_6

$$\begin{array}{ccc} X_{47322} & X_{16303} & X_{6} \\ X_{5112} & X_{468} & X_{1316} \\ X_{32113} & X_{16334} & X_{2453} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{74}

$$\begin{array}{ccc} X_{47323} & X_{47146} & X_{74} \\ X_{47347} & X_{468} & X_{36164} \\ X_{32111} & X_{47148} & X_{477} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{110}

$$\begin{array}{ccc} X_{47324} & X_{47148} & X_{110} \\ X_{47348} & X_{468} & X_{7471} \\ X_{3580} & X_{47146} & X_{476} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{111}

$$\begin{array}{ccc} X_{5913} & X_{16317} & X_{111} \\ X_{47349} & X_{468} & X_{36168} \\ X_{47325} & X_{47350} & X_{2770} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{115}

$$\begin{array}{ccc} X_{187} & X_{230} & X_{115} \\ X_{36180} & X_{468} & X_{14120} \\ X_{47326} & X_{16320} & X_{5099} \end{array}$$

Rectangle with center $(0,0)$ and vertex X_{115}

$$\begin{array}{ccc} X_{47327} & X_{11657} & X_{125} \\ X_{47351} & X_{468} & X_{3154} \\ X_{1495} & X_{16319} & X_{3258} \end{array}$$

6. Crossdifferences

The crossdifference of points $P = p : q : r$ and $U = u : v : w$ is ([4], Glossary) the isogonal conjugate of the trilinear pole of the line PU. In barycentrics, this point, denoted by $D(P,U)$, is given by

$$D(P,U) = a^2(qw - rv) : b^2(ru - pw) : c^2(pv - qu). \tag{36}$$

The representation (36) is much simpler than the representation in Euler coordinates, so that certain remarkable facts about $D(P,U)$ are more readily proved in barycentrics. Such results are shown here as examples:

Example 6.1. *If $P = (x_1, 0)$ and $U = (x_2, 0)$ are distinct points on the x-axis (the Euler line), then $D(P,U) = X_{647}$, on the y-axis. Coordinates for X_{647} are shown in Table 2.*

Example 6.2. *If $P = (0, y_1)$ and $U = (0, y_2)$ are distinct points on the y-axis (the orthic axis), then $D(P, U) = X_3$, on the x-axis.*

Example 6.3. *If $P = (x, 0)$ and $U = (0, y) = (0, 0)$ are points, the first on the Euler line and the second on the orthic axis, then $D(P, U) = X_{468}$.*

Example 6.4. *If $P = (x, 0)$ and $U = (0, y) = (0, 0)$ are points on the line X_2X_{647}, then $D(P, U) = X_{237}$, on the x-axis.*

7. Shinagawa coefficients

Shinagawa coefficients for triangle centers on the Euler line were introduced in 2015; see the introduction to ETC [4]. Their connection to Euler coordinates is presented here. Suppose that X is a triangle center on the Euler line, with normed barycentrics given by $X = (x, y, z)$, where $x = f(a, b, c)$, so that $y = f(b, c, a)$ and $z = f(c, a, b)$. The Shinagawa coefficients of X are the ordered pair (g_X, h_X) of functions of a, b, c such that

$$X = g_X S^2 + h_X S_B S_C ::,$$

and the Euler coordinates of X are $(x, 0)$, where

$$x = \frac{(E+F)g_x + 3Fh_x}{3g_x + h_x}.$$

See [8] for tables of Shinagawa coefficients.

8. The transformation T

Suppose that L is a line and that $P = (p, q, r)$ and $U = (u, v, w)$ are distinct points on L. Then the point $P - U$ is given by the following combo and (not normed) barycentrics

$$P - U = p - u : q - v : r - w. \tag{37}$$

It may be tempting to apply (19) to (37) in order to obtain Euler coordinates (x, y) for points on $\mathcal{L}^\infty$. However, the transformation formulas (19) and (20) depend on the underlying coordinates, $p - u, q - v, r - w$ being normed barycentrics, but they are not, since

$$(p - u) + (q - v) + (r - w) = 1.$$

Nevertheless, we can, and do, *define* a point using (19) and (20). Indeed, this point, which we denote by $T(P, U)$, is the point (x, y) on L that has directed distance

$$\frac{\sqrt{S^2(x_P - x_U)^2 + (y_P - y_U)^2}}{NS}$$

from $(0, 0)$; explicity, $T(P, U) = (x, y)$, where

$$x = S_A(p - u) + S_B(q - v) + S_C(r - w); \tag{38}$$

$$y = S_A(S_B - S_C)(p - u) + S_B(S_C - S_A)(q - v) + S_C(S_A - S_B)(r - w). \tag{39}$$

It must be understood that the transformation T is defined on pairs of points, (P, U), not on individual points $P - U$. A formal treatment follows. Define a relation $\sim$ on the set of all pairs (P, U) by the following statement:

$$(P, U) \sim (P', U') \text{ if } PU = P'U';$$

that is, the line through P and U is also the line through P' and U'. Then $\sim$ is an equivalence relation [10], and its equivalence classes are the lines in the plane of ABC (excluding $\mathcal{L}^\infty$). For each equivalence class, such as the Euler line, any member of the class, such as (X_2, X_3), or (X_4, X_2), or (X_{1113}, X_{1114}) can be chosen as a class representative. A simple analogy follows: on the set of pairs (x, y) of real numbers, define an equivalence relation $\simeq$ as follows:

$$(x, y) \simeq (x', y') \text{ if } x - y = x' - y'.$$

Here, the equivalence classes are the real numbers. For example, the number 2 represents the class that contains $(5, 3), (0, -2)$, and $(19, 17)$. One can now define, for example, a transformation t on an equivalence class by $t(x, y) = x^2 + y^2$; the point of this discussion is now clear: t is defined on the *pairs* (x, y), not on the number $x - y$.

It follows from the definition of triangle center that if P and U are triangle centers, then $T(P, U)$ is a triangle center. Table 4 shows several examples of $T(P, U)$ for the case that L is the Euler line, typified by $X_2 - X_3$, as follows: X_2 has Shinagawa coefficients $(1/3, 0)$ and X_3 has Shinagawa coefficients $(1/2, -1/2)$, so that $X_2 - X_3$ is represented by

$$x = (E + F)/3 - (E/2 - F) = -E/6 + 4F/3.$$

TABLE 4. $T(P, U)$ for selected points P and U on the Euler line

$P - U = X_{30}$	$T(P, U)$, Euler coordinates	$T(P, U)$, barycentrics
$X_2 - X_3$	$(-E/6 + 4F/3, 0)$	see X_{47332}
$X_3 - X_2$	$(E/6 - 4F/3, 0)$	see X_{47333}
$X_2 - X_4$	$(E/3 - 8F/3, 0)$	see X_{47031}
$X_4 - X_2$	$(-E/3 + 8F/3, 0)$	see X_{47310}
$X_2 - X_5$	$(E/12 - 2F/3, 0)$	see X_{18579}
$X_5 - X_2$	$(-E/12 + 2F/3, 0)$	see X_{47334}
$X_3 - X_5$	$(E/4 - 2F, 0)$	see X_{47335}
$X_5 - X_3$	$(-E/4 + 2F, 0)$	see X_{47336}
$X_4 - X_5$	$(-E/4 + 2F, 0)$	see X_{47336}
$X_5 - X_4$	$(E/4 - 2F, 0)$	see X_{47335}
$X_2 - X_{23}$	$(4E/3 + 4F/3, 0)$	see X_{47311}
$X_{23} - X_2$	$(-4E/3 - 4F/3, 0)$	see X_{47312}
$X_4 - X_{23}$	$(E + 4F, 0)$	see X_{47339}
$X_{23} - X_4$	$(-E - 4F, 0)$	see X_{47340}
$X_5 - X_{23}$	$(5E/4 + 2F, 0)$	see X_{47341}
$X_{23} - X_5$	$(-5E/4 - 2F, 0)$	see X_{47342}

In Table 4, note that X_{47335} occurs twice, indicative of the fact that the transformation T is many-to-one; in this regard, see Theorem 9.1.

We return now to (38) and (39) for lines L other than the Euler line. A few examples follow for $L = X_1X_2$, the Nagel line. For all such (P, U), we have

$T(P,U)$ on the line $X_{468}X_{519}$, and also on lines indicated here:

$$\begin{aligned}
T(X_1,X_2) &= X_{47472} = X_1X_{30} \cap X_{515}X_{47310} \cap X_{517}X_{47333}\\
T(X_1,X_8) &= X_{47489} = X_{23}X_{145} \cap X_{30}X_{7982} \cap X_{517}X_{47308} \cap X_{518}X_{47470}\\
T(X_1,X_{10}) &= X_{47491} = X_1X_{858} \cap X_{30}X_{4301} \cap X_{515}X_{47309} \cap X_{516}X_{47469}\\
T(X_2,X_8) &= X_{47493} = X_1X_{47097} \cap X_{30}X_{944} \cap X_{517}X_{47031}\\
T(X_2,X_1) &= X_{47488} = X_{30}X_{40} \cap X_{515}X_{47031} \cap X_{517}X_{47332}\\
T(X_8,X_1) &= X_{47490} = X_8X_{858} \cap X_{30}X_{4677} \cap X_{517}X_{47309}\\
T(X_{10},X_1) &= X_{47492} = X_8X_{23} \cap X_{30}X_{4669} \cap X_{515}X_{47308} \cap X_{517}X_{47336}\\
T(X_8,X_2) &= X_{47494} = X_8X_{30} \cap X_{517}X_{47310}
\end{aligned}$$

Examples for $L = X_3X_6$, the Brocard axis, include the following, which all lie on the line $X_{468}X_{511}$:

$$\begin{aligned}
T(X_3,X_6) &= X_{47468} = X_{30}X_{599} \cap X_{524}X_{32110} \cap X_{542}X_{47031}\\
T(X_3,X_{182}) &= X_{47569} = X_{30}X_{141} \cap X_{524}X_{1511} \cap X_{542}X_{47333}\\
T(X_3,X_{187}) &= X_{47570} = X_{30}X_{114} \cap X_{512}X_{46990} \cap X_{842}X_{858}\\
T(X_6,X_3) &= X_{47571} = X_6X_{30} \cap X_{113}X_{524} \cap X_{517}X_{47506} \cap X_{518}X_{47471}\\
T(X_{15},X_3) &= X_{47575} = X_{13}X_{15} \cap X_{403}X_{621} \cap X_{531}X_{47332}\\
T(X_{16},X_3) &= X_{47576} = X_{14}X_{16} \cap X_{403}X_{622} \cap X_{530}X_{47332}\\
T(X_{182},X_3) &= X_{47581} = X_{30}X_{182} \cap X_{524}X_{47334} \cap X_{542}X_{47332}
\end{aligned}$$

Further examples are listed in [4]; see the preamble just before X_{47488}.

As noted just above, if P and U are on the Nagel line, then $T(P,U)$ is collinear with X_{468} and X_{519}, and if P and U are on the Brocard axis, then $T(P,U)$ is collinear with X_{468} and X_{511}. These two facts exemplify the next theorem.

Theorem 8.1. *If P and U are distinct points, then $T(P,U)$ is collinear with X_{468} and $P - U$.*

Proof. Let $P = (p,q,r)$ and $U = (u,v,w)$, and form $(x,y) = T(P,U)$ using (38) and (39). Then convert (x,y) to barycentrics $t_1 : t_2 : t_3$ using (21). Let D be the determinant having rows (t_1,t_2,t_3), $(p-u, q-v, v-w)$ and

$$\left(\frac{b^2+c^2-2a^2}{b^2+c^2-a^2}, \frac{c^2+a^2-2b^2}{c^2+a^2-b^2}, \frac{a^2+b^2-2c^2}{a^2+b^2-c^2}\right), \text{ or, equivalently,}$$
$$\big((E+F)S_BS_C - 3FS^2, (E+F)S_CS_A - 3FS^2, (E+F)S_AS_B - 3FS^2\big).$$

Then $D = 0$. □

Corollary 8.1. *The lines PU and $X_{468}T(P,U)$ are parallel.*

Proof. The two lines meet in $P - U$, on $\mathcal{L}^\infty$. □

In Figure 4, aside from the parallel lines recognized in Corollary 8.1, another pair of parallel lines appear; these are described in Theorem 8.2

Theorem 8.2. *If P and U are distinct points, then the lines $PT(P,U)$ and UX_{468} are parallel.*

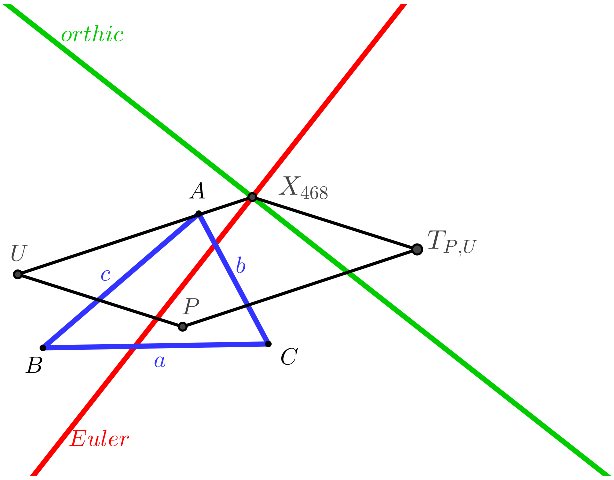

FIGURE 4. $T(P,U)$, vertex of a parallelogram

Proof. Continuing with the notation in the proof of Theorem 8.1, the line $PT(P,U)$ is given by

$$(qt_3 - rt_2)\alpha + (rt_1 - pt_3)\beta + (pt_2 - qt_1)\gamma = 0.$$

This line meets $\mathcal{L}^\infty$, given by line $\alpha + \beta + \gamma = 0$, in the point

$$p(t_2 + t_3) - t_1(q + r) : q(t_3 + t_1) - t_2(r + p) : r(t_1 + t_2) - t_3(p + q). \tag{40}$$

Next, let

$$(h_1, h_2, h_3) = \left(\frac{b^2 + c^2 - 2a^2}{b^2 + c^2 - a^2}, \frac{c^2 + a^2 - 2b^2}{c^2 + a^2 - b^2}, \frac{a^2 + b^2 - 2c^2}{a^2 + b^2 - c^2}\right).$$

Then the line UX_{468} is given by $g_1\alpha + g_2\beta + g_3\gamma = 0$, where

$$(g_1, g_2, g_3) = (vh_3 - wh_2, wh_1 - uh_3, uh_2 - vh_1),$$

which meets $\mathcal{L}^\infty$ in the point $g_3 - g_2 : g_1 - g_3 : g_2 - g_1$. It is easy to check (computer output) that this point is (40). □

9. PROPERTIES OF THE TRANSFORMATION T

We begin with a trivial observation: $T(U,P)$ is the reflection of $T(P,U)$ in the origin, $(0,0) = X_{468}$. Next, we have a result that accounts for duplicates in column 3 of Table 4, and other such duplicates.

Theorem 9.1. *The transformation T is many-to-one. Specifically, suppose that $P = (p,q,r)$ and $U = (u,v,w)$ are points. If $U' = (u',v',w')$ is a point distinct from U, then*

$$T(P,U) = T(P',U'),$$

where

$$\begin{aligned} P' &= P - U + U' \\ &= u(q+r)(v'+w') - p(2u'+v'+w')(v+w) - u'\big(pu + (q+r)(v+w)\big) :: . \end{aligned}$$

Likewise,

$$T(P,U) = T(U',P'),$$

where

$$\begin{aligned} P' &= -P + U + U' \\ &= p(v+w)(v'+w') - u(2u'+v'+w')(q+r) - u'\big(pu + (q+r)(v+w)\big) :: . \end{aligned}$$

Proof. Starting with P and U, and then P' and U', we apply (38) and (39), using, for example, Programs 1 or 2 in the Appendix to produce $T(P,U)$ and $T(P',U')$. We then observe that they are equal, and likewise for $T(U',P')$. □

The previously mentioned examples of points $T(P,U)$ have relatively elaborate barycentrics, when compared to the barycentrics for P and U. It is natural to ask if $T(P,U)$ can possibly be "short". For example, does there exist a pair (P,U) such that $T(P,U) = G$? This question is analogous to asking for a solution to the simple equation $y - x = 2$. As there are many pairs (x,y) that comprise the solution, the problem can be recast thus: for given y, solve for x, and for given x, solve for y. Following this approach, the next two theorems show, for example, that the equations $T(P,U) = G$ and $T(U,P) = G$ both have many solutions.

Theorem 9.2. *Suppose that $P = p : q : r$ and $X = x : y : z$. The solution $U = u : v : w$ of the equation $T(P,U) = X$ is given by*

$$U = f_1 x + f_3(y+z)p + \big(f_2 x + f_1(y+z)\big)(q+r) :: , \tag{41}$$

where

$$\begin{aligned} f_1 &= (2a^2 - b^2 - c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2); \\ f_2 &= (-a^2 + b^2 + c^2)\big(a^2(a^2 - b^2 - c^2) - 3(b^2 - c^2)^2\big); \\ f_3 &= 6a^6 + 5b^6 + 5c^6 - 5(a^4 - b^2c^2)(b^2 - c^2) - 6a^2(b^4 + c^4) + 16a^2b^2c^2. \end{aligned}$$

Proof. As in the proof of Theorem 9.1, proofs of Theorems 9.2 and 9.3 follow from (38) and (39). □

For a second representation of U in (41), suppose that P, X, and U are given by normed barycentrics (p,q,r), (x,y,z), and (u,v,w), and note that the first normed barycentric of X_{468} is given by

$$x_{468} = \frac{(E+F)S_B S_C - 3FS^2}{N^2 S^2}. \tag{42}$$

Then

$$u = p - x + x_{468},$$

and v and w are determined cyclically.

Theorem 9.3. *Suppose that $P = p : q : r$ and $X = x : y : z$. The solution $P = p : q : r$ of the equation $T(P,U) = X$ is given by*

$$P = f_4 x - f_2(y+z)u + \big(f_2 x + f_1(y+z)\big)(v+w) :: , \tag{43}$$

where f_1 and f_2 are as in Theorem 9.2 and

$$f_4 = 6a^6 + 7b^6 + 7c^6 - 7(a^4 - b^2c^2)(b^2 - c^2) - 6a^2(b^4 + c^4) + 20a^2b^2c^2.$$

Proof. (See the proof of Theorem 9.2.) □

For a second representation of P in (42), we continue with the notation used in connection with (42):

$$p = u + x - x_{468},$$

and q and r are determined cyclically. This follows from the parallelism in Corollary 8.1, expressed as $P - X = U - X_{468}$.

To illustrate Theorems 9.2 and 9.3, the solution P of the equation $T(P, I) = G$ is the point

$$\begin{aligned} X_{47593} &= X_1X_{30} \cap X_{468}X_{551} \\ &= I + G - X_{468} \text{ (combo)} \\ &= \prec a \succ (E + F)(S^2 - 3S_BS_C) + 3aN^2S^2 ::, \end{aligned}$$

and the solution U of $T(I, U) = G$ is

$$\begin{aligned} X_{47472} &= X_1X_{30} \cap X_{468}X_{551} \\ &= I - G + X_{468} \\ &= \prec a \succ (E + F)(S^2 - 3S_BS_C - 3aN^2S^2 :: . \end{aligned}$$

10. The orthic axis

Although there is an extensive literature on the subject of the Euler line, the orthic axis, defined as the trilinear polar of the orthocenter, is much less well known. Because of its importance as the y-axis in the Euler coordinate system, several computer-discovered properties of the orthic axis are presented here.

10.1. $D(X)$ and crossdifferences.

Suppose that $X = x : y : z$, and define

$$D(X) = S_Bz - S_Cy :: \quad \text{and} \quad D^*(X) = S_By - S_Cz :: .$$

Then $D(X)$ lies on the orthic axis, whereas $D^*(X)$ lies on $\mathcal{L}^\infty$. Moreover, $D(X)$ is the crossdifference of every pair of points on the line $X_3\hat{X}$, where $\hat{X}$ denotes the isogonal conjugate of the isotomic conjugate of X. The appearance of (i, j, k) in the following list means that the crossdifference of every pair of distinct points on the line X_iX_j is X_k, on the orthic axis:

$$(3, 512, 230), (3, 1510, 231), (3, 525, 232), (3, 30, 647), (3, 523, 3003).$$

10.2. Orthic axis and nine-point circle.

The G-Ceva conjugate of the orthic axis is the nine-point circle, so that the G-Ceva conjugate of the nine-point circle is the orthic axis. In the following list, the appearance of (i, j) means that X_i, on the nine-point circle, is identical to the G-Ceva conjugate of X_j, where X_j lies on the orthic axis:

$$\begin{aligned} &(11, 650), (113, 3003, (114, 230), (115, 523), (125, 647), \\ &(128, 231), (132, 232), (1560, 468), (1566, 676) \end{aligned}$$

10.3. **Four special circles.** There are four named circles with centers on the orthic axis, described in Table 5.

TABLE 5. Four circles centered on the orthic axis

Circle	Center	References
Stevanović circle	X_{650}	MathWorld [9] and X_{47199}
Dao-Moses-Telv circle	X_{1637}	X_{6103} (preamble) and $X_{47200} - X_{47214}$
Moses radical circle	X_{647}	X_{6111} (preamble) and $X_{47215} - X_{47225}$
Moses-Parry circle	X_{2492}	X_{8428} (preamble) and $X_{47226} - X_{27236}$

10.4. **A construction.** As indicated in Figure 5, the point X_{647} on the orthic axis is notably related to several other triangle centers:

$$X_{647} = \text{anticomplement of } X_{850};\ \text{i.e., } G \text{ trisects the segment} X_{647}X_{850};$$
$$X_{47004} = \text{reflection of } X_{647} \text{ in } X_{468};$$
$$X_{23} = X_{850}X_{47704} \cap \text{Euler line.}$$

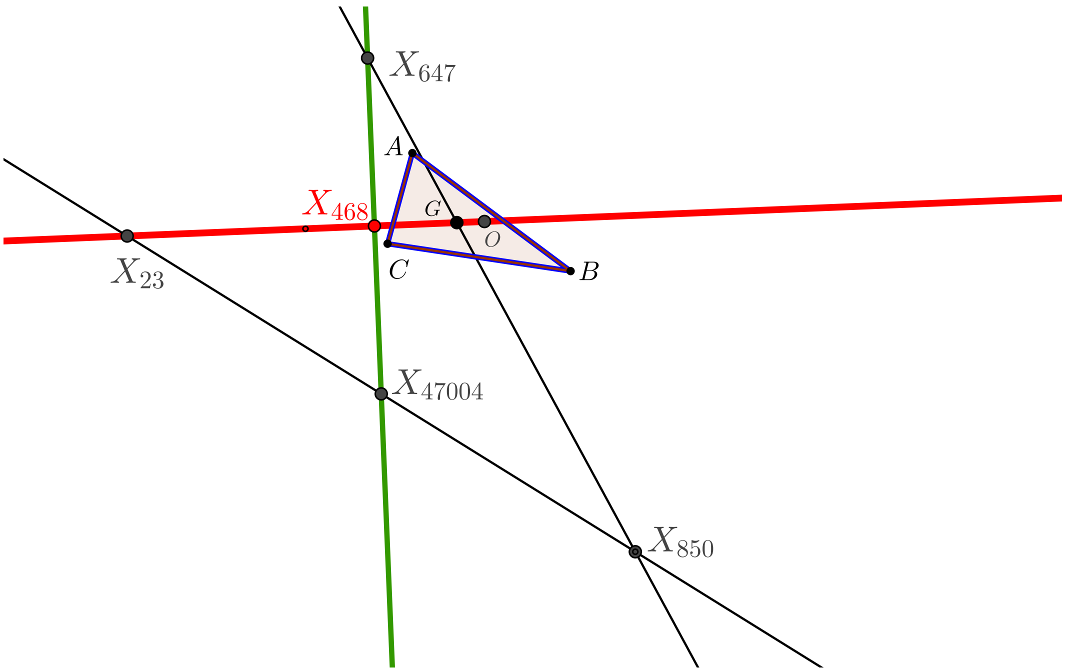

FIGURE 5. X_{647} and related triangle centers.

10.5. **Glide-rotations.** The term *glide-rotation* applies to points on the x-axis and y-axis. In both cases, the rotation is in the positive direction; i.e., the direction determined by rotating the positive x-axis about $(0,0)$ to the positive y-axis. (As noted in Section 2, this direction is not necessarily counterclockwise.) The glide-rotation of a point $(x,0)$ is defined as the point $(0,Sx)$, and the glide-rotation of a point $(0,y)$ is $(y/S,0)$. Thus, glide-reflection maps the entire x-axis onto the entire y-axis. For example, P_{207} in [3] is the glide-rotation of $X_4 = (0,3F)$. The bicentric mate of P_{207} is U_{207}, which is the glide-rotation of $(0,-3F) = X_{10295}$. The reverse kind of glide-rotation transforms the bicentric pair

$$(0, \frac{(E-8F)S^2}{E+F}) \text{ and } (0, -\frac{(E-8F)S^2}{E+F})$$

into X_{47614} and X_{47615}.

Examples of glide-rotations of points $(0, kSS_\omega)$ and $(0, kS^2)$, for selected values of k, to points on the Euler line are listed in the preamble [4] just before X_{47629}.

11. appendix

In the following Mathematica programs, the lower case symbols e and f represent E and F defined in Section 1. Also, S^2 is coded as s2, and S_A, S_B, S_C as SA, SB, SC.

Program 1: Coordinate conversion from barycentric to Euler, using Conway notation

```
s2 = SB*SC + SC*SA + SA*SB;
e = (SB + SC) (SC + SA) (SA + SB)/s2; f = SA*SB*SC/s2 ;
be[{p_, q_, r_}] :=
  Factor[{(SA*p + SB*q + SC*r)/(p + q + r), (SA*(SB - SC)*p +
       SB*(SC - SA)*q + SC*(SA - SB)*r)/(p + q + r)}];
be[{1, 1, 1}]
```

Output of Program 1:

```
{1/3 (SA + SB + SC), 0}
```

Program 2: Coordinate conversion from barycentric to Euler, using a, b, c notation

```
{SA, SB, SC} = {(b^2 + c^2 - a^2)/2, (c^2 + a^2 - b^2)/
    2, (a^2 + b^2 - c^2)/2};
s2 = SB*SC + SC*SA + SA*SB; e = (SB + SC) (SC + SA) (SA + SB)/s2; f =
 SA*SB*SC/s2 ;
be[{p_, q_, r_}] :=
  Factor[{(SA*p + SB*q + SC*r)/(p + q + r), (SA*(SB - SC)*p +
       SB*(SC - SA)*q + SC*(SA - SB)*r)/(p + q + r)}];
be[{1, 1, 1}]
be[{a, b, c}]
```

Output of Program 2:

```
{1/6 (a^2 + b^2 + c^2), 0}
{-((a^3 - a^2 b - a b^2 + b^3 - a^2 c - b^2 c - a c^2 - b c^2 + c^3)/(
  2 (a + b + c))), 1/2 (a - b) (a - c) (b - c) (a + b + c)}
```

Program 3: Coordinate conversion from Euler to barycentric, using a, b, c notation

```
{SA, SB, SC} = {(b^2 + c^2 - a^2)/2, (c^2 + a^2 - b^2)/
    2, (a^2 + b^2 - c^2)/2};
s2 = SB*SC + SC*SA + SA*SB; e = (SB + SC) (SC + SA) (SA + SB)/s2; f =
 SA*SB*SC/s2 ;
eb[{x_, y_}] := Factor[{
    (e + f)*SB*SC - 3*f*s2 - (3 SB*SC - s2)*x + (SB - SC)*y,
    (e + f)*SC*SA - 3*f*s2 - (3 SC*SA - s2)*x + (SC - SA)*y,
    (e + f)*SA*SB - 3*f*s2 - (3 SA*SB - s2)*x + (SA - SB)*y}];
t = eb[{(e - 8 f)/3, 0}]  (* X(47031) *)
Numerator[Factor[t[[1]]]]
Numerator[Factor[t[[2]]]]
Numerator[Factor[t[[3]]]]
```

Output of Program 3: (See X_{47031}.)

Program 4:: The point T (P, U) from points P = p : q : r and U = u : v : w, coded as p, q, r and u, v, w

```
{p, q, r} = {1, 1, 1};
{u, v, w} = {SB SC, SC SA, SA SB};
s2 = SB*SC + SC*SA + SA*SB; e = (SB + SC) (SC + SA) (SA + SB)/s2; f =
 SA*SB*SC/s2 ;
{p1, q1, r1} = {p, q, r}/(p + q + r); {u1, v1,
  w1} = {u, v, w}/(u + v + w);
x = SA*(p1 - u1) + SB*(q1 - v1) + SC*(r1 - w1);
y = SA*(SB - SC) (p1 - u1) + SB (SC - SA)*(q1 - v1) +
   SC (SA - SB)*(r1 - w1);
{t1, t2, t3} =
  Factor[{(e + f)*SB*SC - 3*f*s2 - (3 SB*SC - s2)*x + (SB - SC)*y,
    (e + f)*SC*SA - 3*f*s2 - (3 SC*SA - s2)*x + (SC - SA)*y,
    (e + f)*SA*SB - 3*f*s2 - (3 SA*SB - s2)*x + (SA - SB)*y}];
t1 (* 1st barycentric of T(P,U) using Conway notation *)
{SA, SB, SC} = {(b^2 + c^2 - a^2)/2, (c^2 + a^2 - b^2)/
    2, (a^2 + b^2 - c^2)/2};
{t4, t5, t6} =
  Factor[{(e + f)*SB*SC - 3*f*s2 - (3 SB*SC - s2)*x + (SB - SC)*y,
    (e + f)*SC*SA - 3*f*s2 - (3 SC*SA - s2)*x + (SC - SA)*y,
    (e + f)*SA*SB - 3*f*s2 - (3 SA*SB - s2)*x + (SA - SB)*y}];
t4 (* 1st barycentric of T(P,U)using a,b,c notation *)
```

Output of Program 4: (See Table 4 and the lists just after Table 4.)

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